# Quotients of Spheres and the Tutte Polynomial

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# Why Study Quotients of Spheres?

Let a compact group $G$ act isometrically on a Riemannian manifold $M$.
What does $M/G$ look like?
Is it a topological manifold?

Let $x$ be a point in $M$ with tangent space $T_xM$. Consider $S_x$, the unit tangent sphere at $x$ in $T_xM$.

Then the subgroup $G_x \subseteq G$ that fixes $x$ acts on $S_x$, and $M/G$ can only be a topological manifold if the quotient $S_x/G_x$ is at least a homology sphere.

# What is a Matroid?

A matroid $M$ is a pair $(E, \mathcal{I})$. The edge set $E$ is a finite set, and $\mathcal{I}(M) \subseteq \mathcal{P}(E(M))$ denotes the independent subsets of $E$ that respect the following axioms:

- $\emptyset \in \mathcal{I}$
- If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$
- If $A, B \in \mathcal{I}(M)$ and $|A| > |B|$, then $\exists e \in A$ such that $B \cup e \in \mathcal{I}$

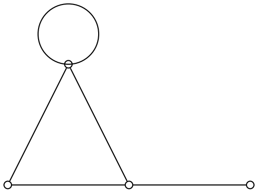
An element $e \in M$ is a *loop* if it is no independet set.
An element $e \in M$ is a *coloop* if it is in every basis (maximal independent set)

# Examples of Matroids

The columns of a matrix are the edge set of a matroid; subsets are independent iff they are linearly independent.

The edges of a graph can be taken as the edge set of a matroid; sets of edges are independent if and only if they contain no cycles.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$



# The Tutte Polynomial

The Tutte Polynomial $T(M; x, y)$ is defined recursively by:

- $T$(a single coloop; x,y) $= x$
- $T$ (a single loop; x,y) $= y$
- If $e$ is a loop or coloop,
  $T(M; x, y) = T(e; x, y)T(M - e; x, y)$
- If $e$ is neither,
  $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$

Many important graph and matroid invariants satisfy these recursions, and are thus evaluations of the Tutte polynomial

# A History Lesson: Lens Spaces

Let $\bar{1}$ denote the generator of $\mathbb{Z}_k$
Let $k, m \in \mathbb{Z}$ such that **gcd(k,m)=1**

A 3-dimensional lens space $L(k, m)$ is a quotient of $S^3 \subseteq \mathbb{C}^2$ by $\mathbb{Z}_k$ given by $\bar{1} \cdot (z_1, z_2) = (e^{2\pi \imath/k} \cdot z_1, e^{2\pi \imath m/k} \cdot z_2)$.

The action rotates the two unit circles on the planes $z_1 = 0$ and $z_2 = 0$. The action on the rest of the spheres is determined by these rotations, since $S^3 = S^1 * S^1$

Lens spaces were introduced by Tietze in 1908. In 1919, Alexander showed that $L(5, 1)$ and $L(5, 2)$ are not homeomorphic despite having the same homology and fundamental group.

# Quotients By Cyclic Groups

Let $X$ be the quotient of $S^{2n-1}/\mathbb{Z}_{p^k}$ where $\bar{1}$ acts by rotating the unit circles by: $2\pi/p^{a_1}$, $2\pi/p^{a_2}, \cdots$ $2\pi/p^{a_n}$ where $a_1 \leq a_2 \leq \cdots \leq a_n = k$ Then:

$H_{2n-1}(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^k}$
$H_{2n-2}(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_{n-1}}}$
$H_{2n-3}(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_{n-1}}}$
$H_{2n-4}(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_{n-2}}}$
$H_{2n-5}(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_{n-2}}}$
$\vdots$
$H_2(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_1}}$
$H_1(X;\mathbb{Z}_{p^k}) = \mathbb{Z}_{p^{a_1}}$

Theorem proven by Willson in 1976; his proof was more general and applied to $\mathbb{Z}_{p^k}$ homology spheres.

# Ed's Thesis

In 1999, Swartz found a one-to-one correspondence between isometry classes of quotient $S^n/\mathbb{Z}_2^r$ and *binary matroids*. Furthermore, he determined the reduced Poincaré polynomial of the quotient to be $\tilde{P}(X;\mathbb{Z}_2) \cong t^{r-1}T(M_X;0,t)$

More generally, $H_i(S^{2n-1}/(\mathbb{Z}_p)^r;\mathbb{Z}_p) \cong t^{r-1}T(2M;0,t)$, though the matroid correspondence is not one-to-one for odd primes. In particular, $L(5,1)$ and $L(5,2)$ are a counterexample.

Swartz also computes the homology of the *free part* of the action, i.e. the quotient of the subset of the sphere on which the action is free as: $\tilde{P}(X^f;\mathbb{Z}) = t^{r-1}T(2M,1,t)$

# The Correspondence of Swartz for $\mathbb{Z}_2^r$

Given a subgroup $G \cong \mathbb{Z}_2^r \subseteq O(n)$, we can choose a set of generators $\gamma_1, \gamma_2, \cdots \gamma_r$ for $G$. We can choose these $\gamma_i$ to be diagonal matrices: any basis for $G$ can be *simultaneously diagonalized* since $G$ is abelian.

This diagonalization will not affect the geometry of the quotient, since conjugate subgroups of $O(n)$ yield isometric quotients

All the $\gamma_i$ have $\pm 1$ entries on their diagonals. We can rewrite each diagonal as a row vector of length $n$, converting 1's to 0's and -1's to 1's. Combining these rows into a $r \times n$ matrix gives a representation of the associated matroid.

# Example: $S^2/\mathbb{Z}_2$

| Generator of $\mathbb{Z}_2$ | Matroid | $T(M_x; 0, t)$ | Quotient |
|---|---|---|---|
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ | $0$ | Closed Hemisphere |
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ | $t^2$ | Football |
| $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ | $t^2 + t$ | $\mathbb{R}P^2$ |

# A Torus Action on an Odd-Dimensional Sphere

The Torus can be decomposed $T^r = T_1^1 \times T_2^1 \times \cdots T_r^1$ Since it is acting on the sphere, we know $T^r \subseteq O(2n-1)$; every element of $T^r$ is an orthonormal matrix.

Recall $S^{2n-1} = S_1^1 * S_2^1 * \cdots * S_n^1$, where * denotes the topological join of spaces

Consider the action of a single circle $T_i^1$ of the torus, restricted to any one of the circles in the join $S^{2n-1}$. This action, $T_i \curvearrowright S_j$, is given by $e^{\imath\alpha} \cdot e^{\imath\beta} = e^{\imath(k_{ij}\alpha+\beta)}$ for some $k_{ij} \in \mathbb{Z}$.

Let $A$ be a matrix that describes the entire action $T^r \curvearrowright S^{2n-1}$, with i-jth entry $k_{ij}$, which describes the action of $T_i$ on $S_j$.

# Why Use the Matroid?

We have a matrix with $r$ rows, corresponding to the $r$ cicles of $T^r$ and $n$ columns, one for each $S^1$ in the decomposition of $S^{2n-1}$. The $ij-th$ entry describes the action $T_i \curvearrowright S_j$

| | |
|---|---|
| Column switching | Changing the ordering of the components $S^1_j$ |
| Row switching | Changing the ordering of the $T_i$ in $T^r$ |
| Adding one row to another | Choosing a new basis for $T^r$ |

These operations all preserve the matroid structure, as well as the geometry of the quotient.

# Learning from the Matroid

Proposition 1: If $M$ contains a coloop, then $X = S^{2n-1}/T^r$ is contractible.

Proof: If $e \in M$ is a coloop, it acts independently of the other elements.
So $S^{2n-1}/G = S^{2n-3}/(G/<\gamma_e>) * S^1/<\gamma_e> =$
$S^{2n-3}/(G/<\gamma_e>) *$ (a point)

Proposition 2: Let $X = S^{2n-1}/T^r$ and let $M_X$ be the corresponding matroid. If $M_X$ contains a loop $e_j$, then
$X \cong S_j * X_{M-e_j}$;

# The Homology and its Generators

Theorem (H., Swartz): Let $X \cong S^{2n-1}/T^r$ be a quotient by a linear effective action, with associated matroid $M_X$. Then $\tilde{P}(X;\mathbb{Z}) = t^{r-1}T(M_X;0,t^2)$

Corollary: There is a one-to-one correspondence between generators of homology in dimension $r+2m-1$ and bases of $M$ of internal activity 0 and external activity $m$.

# The Singular Set

Given a quotient space $X = Y/G$, the singular set $\mathcal{S}$ of the action is the image in the quotient space of the points whose isotropy groups are infinite.

Theorem (H., Swartz): For $X \cong S^{2n-1}/T^r$ a quotient of an effective linear action, with associated matroid $M_X$, $\tilde{P}(\mathcal{S}(X); \mathbb{Z}) \cong t^{r(M_X)-2}(T(M_X; 1, t^2) - T(M_X; 0, t^2))$

Sketch of proof: The singular set is an arrangement of images of subspheres $S^1_{i_1} * S^1_{i_2} * \cdots$ of $S^{2n-1}$. Each subsphere has a corresponding flat of the matroid that fixes it; it can be shown that the lattice of the singular arrangement is the dual of the lattice of flats of $M_X$. The homology can be computed using The Wedge Lemma of Ziegler and Živaljević

# Quotients by Finite Abelian Groups

We need a different matrix/matroid to describe the action for each $p^i$ where $1 \leq i \leq k$ if $k$ is the maximal power of $p$ appearing as an order in the finite group $G$.

To determine $2M_{p^i}$, first take a matrix describing the action. Then rewrite the elements mod $p^i$. If $i \neq 1$, the matrix does not directly represent a matroid since it is not over a field; view the columns as elements of $\mathbb{Z}_{p^i}^n$, and assign a rank function to the set $A$ of columns as the rank of the subgroup generated by $A$ tensored with $\mathbb{Z}_p$.

We get a sequence of matroids $M_{p^k}, M_{p^{k-1}}, \cdots, M_p$, with weak maps between them (i.e. each new matroid may have more dependencies than the last)

# Conjecture for $H_i(S^{2n-1}/G; \mathbb{Z}_{p^k})$ where $G$ is a finite abelian group

We get a corresponding sequence of Tutte polynomials:
$t^{r-1}T(2M_{p^k}; 0, t), t^{r-1}T(2M_{p^{k-1}}; 0, t), \cdots t^{r-1}T(2M_p; 0, t)$

If a copy of $t^i$ appears in an isolated instance in the chain of Tutte polynomials, it corresponds to a $\mathbb{Z}_p$ summand in $H_i(X)$

If a copy of $t^i$ appears in two (but not 3) consecutive polynomials, it corresponds to a $\mathbb{Z}_{p^2}$ summand in $H_i(X)$

$\vdots$

If a summand appears in every Tutte polynomial, it corresponds to a $\mathbb{Z}_{p^k}$ summand in the homology

# An Example of the Conjecture at Work

Consider the action of $\mathbb{Z}_4 \times \mathbb{Z}_4$ on $S^7$ given by the following matrix with entries modulo 4:

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_4 = U_{2,4}$$

Then $t^{r-1}T(2M_4; 0, t) = y^7 + 2y^6 + 3y^5 + 4y^4 + 5y^3 + 2y^2$

If we reduce this matrix modulo 2, we get a new matroid $M_2$, where the last two edges are parallel.

Then $t^{r-1}T(2M_2; 0, t) = y^7 + 2y^6 + 3y^5 + 3y^4 + 3y^3 + y^2$

# An Example of the Conjecture at Work

The quotient of $S^7$ by $\mathbb{Z}_4 \times \mathbb{Z}_4$ represented by: $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$$t^{r-1}T(2M_4; 0, t) = y^7 + 2y^6 + 3y^5 + 4y^4 + 5y^3 + 2y^2$$
$$t^{r-1}T(2M_2; 0, t) = y^7 + 2y^6 + 3y^5 + 3y^4 + 3y^3 + y^2$$

$$H_1(X; \mathbb{Z}_4) = 0$$
$$H_2(X; \mathbb{Z}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$$
$$H_3(X; \mathbb{Z}_4) = (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_4)^3$$
$$H_4(X; \mathbb{Z}_4) = \mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3$$
$$H_5(X; \mathbb{Z}_4) = (\mathbb{Z}_4)^3$$
$$H_6(X; \mathbb{Z}_4) = (\mathbb{Z}_4)^2$$
$$H_7(X; \mathbb{Z}_4) = \mathbb{Z}_4$$

# References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